

GRADIENT FLOWS OF THE ENTROPY FOR JUMP PROCESSES

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ABSTRACT. We introduce a new transportation distance between probability measures on \mathbb{R}^d that is built from a Lévy jump kernel. It is defined via a non-local variant of the Benamou-Brenier formula. We study geometric and topological properties of this distance, in particular we prove existence of geodesics. For translation invariant jump kernels we identify the semigroup generated by the associated non-local operator as the gradient flow of the relative entropy w.r.t. the new distance and show that the entropy is convex along geodesics.

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1. INTRODUCTION

In the last two decades the theory of optimal transportation has found applications to many areas of mathematics such as partial differential equations, geometry and probability. We refer the reader to the monograph [27] for an overview. In particular, optimal transport has proved very useful in the study of diffusion processes. One of the most striking examples is Otto's discovery [18, 24] that many diffusion equations can be interpreted as gradient flows of a suitable free energy functional with respect to the L^2 -Wasserstein distance on the space of probability measures. A prominent example is the heat equation which is the gradient flow of the Shannon entropy. By now, similar interpretations of the heat flow have been established in a variety of settings ranging from Riemannian manifolds to abstract metric measure spaces, see [13, 23, 15, 17, 2].

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The aim of this article is to build a bridge between the theory of jump processes and non-local operators on one hand and ideas from optimal transportation on the other hand. We will give a gradient flow interpretation of the equation

$$\partial_t u = \mathcal{L}u, \quad (1.1)$$

where \mathcal{L} is a non-local operator given by

$$\mathcal{L}u(x) = \int u(y) - u(x) - (y - x) \cdot \nabla u(x) \mathbf{1}_{\{|y-x|<1\}} J(x, dy),$$

with a Lévy measure $J(x, dy)$ for every $x \in \mathbb{R}^d$. Such operators arise as the generators of a pure jump Feller process. For this purpose the Wasserstein distance is not appropriate. The main contribution of this article is thus the construction of a new transportation distance on the space of probability measures that is non-local in nature and allows to interpret equation (1.1) formally as the gradient flow of the relative entropy. We define this distance via a non-local variant of the dynamical characterization of the Wasserstein distance by Benamou and Brenier [7]. A prominent example we will often consider is given by the choice $J_\alpha(x, dy) = c_\alpha |y - x|^{-\alpha-d} dy$ with $\alpha \in (0, 2)$ corresponding to the fractional Laplacian $\mathcal{L} = -(-\Delta)^{\frac{\alpha}{2}}$ which is a pseudo differential operator with symbol $|\xi|^\alpha$. For translation invariant jump kernels such as J_α where the underlying jump process is a Lévy process, we rigorously identify the equation as the gradient flow of the entropy w.r.t. the new distance in the framework of gradient flows in metric spaces developed in [1]. Moreover, we show that the entropy is convex along geodesics.

To motivate our interest in such a link between jump processes and optimal transport, let us highlight two observations.

The gradient flow approach has been used as a powerful tool in the study of many evolution partial differential equations. Already in Otto's original work [24] convexity properties of the entropy functional have been used to derive explicit rates of convergence to equilibrium for the porous medium equation. This approach is also well adapted to the study of functional inequalities, such as logarithmic Sobolev inequalities (see e.g. the famous result by Otto-Villani [25]). Recently, it has been shown that the gradient flow characterization provides a good framework to study stability properties of diffusion processes under changes of the driving potential or the underlying geometry [3], [16].

The regularity theory for elliptic and parabolic equations involving non-local operators is under active development including both analytic and probabilistic approaches (see e.g. [9], [6] and references therein). In a local setting very precise regularity results can be obtained using a lower bound on the Ricci curvature of the operator in the sense of the Bakry-Émery criterion [5]. Equivalently, such curvature information can be encoded into convexity properties of the entropy along Wasserstein geodesics. In fact, geodesic convexity of the entropy has been used as a synthetic notion of a lower Ricci curvature bound for metric measure spaces by Lott-Villani [19] and Sturm [26]. In this sense the approach presented here could be used to define an alternative notion of curvature in the spirit of Lott-Villani-Sturm

that might be more adapted to certain situations than the non-local Γ^2 -calculus. In the discrete setting of finite Markov chains, this approach has already been used in [14] to derive new functional inequalities.

Modifications of the Wasserstein distance have been considered recently by a number of authors. In [12] Dolbeault, Nazaret and Savaré proposed a new class of transport distances based on an adaptation of the Benamou-Brenier formula to give a gradient flow interpretation to a class of transport equations with non-linear mobilities. Very recently, Maas [20] (see also [22], [10] for independent related work by Mielke and Chow et al.) introduced a distance between probability measures on a discrete space equipped with a Markov kernel such that the law of the continuous time Markov chain evolves as the gradient flow of the entropy. Our approach is very similar in spirit to the work of Maas and generalizes it to a certain extent. On the technical side we use an adaptation of the techniques developed in [12] to our non-local setting.

Main results. Let us now discuss the content of this article in more detail. Let $(J(x, \cdot), x \in \mathbb{R}^d)$ be a jump kernel. By this we mean that for all $x \in \mathbb{R}^d$ $J(x, \cdot)$ is a Radon measure on $\mathbb{R}^d \setminus \{x\}$ depending measurably on x . Throughout this text J shall satisfy the following

Assumption 1.1. *For every bounded continuous function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ the mapping*

$$x \mapsto \int f(y)(1 \wedge |x - y|^2)J(x, dy)$$

is again bounded and continuous.

In particular $(J(x, \cdot), x \in \mathbb{R}^d)$ is a so called Lévy kernel (see e.g. [4, Ch. 3.5]). Further let m be a Radon measure on \mathbb{R}^d . We assume that J is reversible w.r.t. m , i.e. the measure $J(x, dy)m(dx)$ is symmetric.

We denote by $\mathcal{P}(\mathbb{R}^d)$ the space of Borel probability measures on \mathbb{R}^d . Given $\mu \in \mathcal{P}(\mathbb{R}^d)$ we define its relative entropy w.r.t. m by

$$\mathcal{H}(\mu) = \int \rho \log \rho \, dm$$

if μ is absolutely continuous w.r.t. m with density ρ and $(\rho \log \rho)_+$ is integrable. Otherwise we set $\mathcal{H}(\mu) = +\infty$.

A non-local transportation distance. Let us first motivate the construction of our new metric by recalling the dynamical characterization of the L^2 -Wasserstein distance. The Benamou-Brenier formula [7] asserts that for two probability densities $\bar{\rho}_0, \bar{\rho}_1$ on \mathbb{R}^d we have

$$W_2^2(\bar{\rho}_0, \bar{\rho}_1) = \inf_{\rho, \psi} \int_0^1 \int |\nabla \psi_t(x)|^2 \rho_t(x) dx dt, \quad (1.2)$$

where the infimum is taken over all sufficiently smooth functions $\rho : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}_+$ and $\psi : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}$ subject to the continuity equation

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \nabla \psi) = 0, \\ \rho_0 = \bar{\rho}_0, \rho_1 = \bar{\rho}_1. \end{cases} \quad (1.3)$$

Here we will define a (pseudo-)metric (i.e. possibly attaining the value $+\infty$) on $\mathcal{P}(\mathbb{R}^d)$ by giving a non-local analogue of formulas (1.2) and (1.3). In order to obtain a metric with the desired properties it is necessary to introduce a function $\theta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying Assumption 2.1 below and to consider the mean $\hat{\rho}(x, y) := \theta(\rho(x), \rho(y))$ of a given density $\rho : \mathbb{R}^d \rightarrow \mathbb{R}$ at different points. We will be mostly interested in the logarithmic mean

$$\theta(s, t) = \frac{s - t}{\log s - \log t} \quad (1.4)$$

but for future use we allow for more generality in the construction. For a function $\psi : \mathbb{R}^d \rightarrow \mathbb{R}$ we will denote by $\bar{\nabla}\psi(x, y) = \psi(y) - \psi(x)$ its discrete gradient. Following the approach of [20] one is led to consider the following ‘distance’. Given probability measures $\bar{\mu}_0 = \bar{\rho}_0 m$ and $\bar{\mu}_1 = \bar{\rho}_1 m$ set

$$\widetilde{\mathcal{W}}(\bar{\mu}_0, \bar{\mu}_1)^2 := \inf_{\rho, \psi} \frac{1}{2} \int_0^1 \int |\bar{\nabla}\psi_t(x, y)|^2 \hat{\rho}_t(x, y) J(x, dy) m(dx) dt, \quad (1.5)$$

where the infimum is now taken over all functions ρ and ψ satisfying the ‘continuity equation’

$$\begin{cases} \partial_t \rho_t + \bar{\nabla} \cdot (\hat{\rho}_t \bar{\nabla} \psi_t) = 0, \\ \rho_0 = \bar{\rho}_0, \quad \rho_1 = \bar{\rho}_1, \end{cases} \quad (1.6)$$

in the sense that for every test function $\varphi \in C_c^\infty(\mathbb{R}^d)$ we have

$$\int \varphi \partial_t \rho_t(x) m(dx) - \frac{1}{2} \int \bar{\nabla} \varphi(x, y) \bar{\nabla} \psi(x, y) \hat{\rho}(x, y) J(x, dy) m(dx) = 0.$$

Instead of addressing the variational problem (1.5) directly we will adopt a measure theoretic point of view and recast it in the more natural relaxed setting of time-dependent families of Radon measures. Let us briefly sketch this approach.

We let $G = \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d : x \neq y\}$ and fix $\gamma(dx, dy) = J(x, dy) m(dx)$. We replace ρ by a continuous curve $t \mapsto \mu_t = \rho_t m$ in $\mathcal{P}(\mathbb{R}^d)$ and ψ_t induces a family of signed Radon measures $\nu_t(dx, dy) = \bar{\nabla} \psi_t(x, y) \hat{\rho}_t(x, y) \gamma(dx, dy)$ on G . The couple (μ, ν) now satisfies the linear equation

$$\begin{cases} \partial_t \mu_t + \bar{\nabla} \cdot \nu_t = 0, \\ \mu_0 = \bar{\mu}_0, \quad \mu_1 = \bar{\mu}_1 \end{cases} \quad (1.7)$$

which we understand in the sense of distributions, i.e. for all test functions $\varphi \in C_c^\infty((0, 1) \times \mathbb{R}^d)$:

$$\int_0^1 \int \partial_t \varphi d\mu_t dt + \frac{1}{2} \int_0^1 \int \bar{\nabla} \varphi(x, y) \nu_t(dx, dy) dt = 0.$$

The quantity to be minimized in (1.5) can now be rewritten as

$$\frac{1}{2} \int_0^1 \int \left| \frac{d\nu_t}{d\gamma}(x, y) \right|^2 \theta \left(\frac{d\mu_t}{dm}(x), \frac{d\mu_t}{dm}(y) \right)^{-1} \gamma(dx, dy) dt.$$

We will define a distance \mathcal{W} by proceeding as follows. To any $\mu \in \mathcal{P}(\mathbb{R}^d)$ we associate two Radon measures on G by setting $\mu^1(dx, dy) = J(x, dy) \mu(dx)$ and $\mu^2(dx, dy) = J(y, dx) \mu(dy)$. Given a Radon measure ν on G we choose

a reference measure σ on G such that $\nu = w\sigma$ and $\mu^i = \rho^i\sigma$, $i = 1, 2$ are all absolutely continuous w.r.t. σ . Then we define the action functional by

$$\mathcal{A}(\mu, \nu) := \frac{1}{2} \int \left| \frac{d\nu}{d\sigma} \right|^2 \theta \left(\frac{d\mu^1}{d\sigma}, \frac{d\mu^2}{d\sigma} \right)^{-1} d\sigma .$$

Assumptions on θ will guarantee that the map $(w, s, t) \mapsto w^2\theta(s, t)^{-1}$ is homogeneous, hence the definition of \mathcal{A} is independent of the choice of σ . Given two measures $\bar{\mu}_0, \bar{\mu}_1 \in \mathcal{P}(\mathbb{R}^d)$ we denote by $\mathcal{CE}_{0,1}(\bar{\mu}_0, \bar{\mu}_1)$ the set of all sufficiently regular solutions (to be made precise in section 3) $(\mu_t, \nu_t)_{t \in [0,1]}$ of the continuity equation (1.7).

Definition. For $\bar{\mu}_0, \bar{\mu}_1 \in \mathcal{P}(\mathbb{R}^d)$ we define

$$\mathcal{W}(\bar{\mu}_0, \bar{\mu}_1)^2 := \inf \left\{ \int_0^1 \mathcal{A}(\mu_t, \nu_t) dt : (\mu, \nu) \in \mathcal{CE}_{0,1}(\bar{\mu}_0, \bar{\mu}_1) \right\} .$$

It is unclear whether \mathcal{W} coincides with $\widetilde{\mathcal{W}}$ defined in (1.5) in full generality. However, we will give a positive answer for the more restricted case of a sufficiently regular translation invariant jump kernel such as J_α (see Proposition 5.8). We can now state the first main result of this article.

Theorem 1.2. \mathcal{W} defines a (pseudo-) metric on $\mathcal{P}(\mathbb{R}^d)$. The topology it induces is stronger than the topology of weak convergence. For each $\tau \in \mathcal{P}(\mathbb{R}^d)$ the set $\mathcal{P}_\tau := \{\mu \in \mathcal{P}(\mathbb{R}^d) : \mathcal{W}(\mu, \tau) < \infty\}$ equipped with the distance \mathcal{W} is a complete geodesic space.

Gradient flow of the entropy. Let us give a short formal argument why equation (1.1) can be seen as the gradient flow of the relative entropy w.r.t. the distance \mathcal{W} if we choose θ to be the logarithmic mean.

In the classical setting many partial differential equations of the form

$$\partial_t \rho - \nabla \cdot (\rho \nabla f'(\rho)) = 0$$

can, at least formally, be seen as the gradient flow of the integral functional $\mathcal{F}(\rho) = \int f(\rho) dm$ w.r.t. the L^2 -Wasserstein distance. Hence in the new geometry determined by the distance $\widetilde{\mathcal{W}}$ via (1.5), (1.6) the gradient flow of the functional \mathcal{F} should be given by the equation

$$\partial_t \rho - \bar{\nabla} \cdot (\bar{\rho} \bar{\nabla} f'(\rho)) = 0 .$$

If we now consider the relative entropy \mathcal{H} we have $f'(r) = 1 + \log r$. Taking into account (1.4) we see that the corresponding gradient flow is given by

$$\partial_t \rho - \bar{\nabla} \cdot (\bar{\nabla} \rho) = 0 ,$$

which is a weak formulation of (1.1). In particular we see that the appearance of the logarithmic mean is necessary in order to account for the fact that the discrete gradient lacks a chain rule.

In the more restricted setting of a translation invariant jump kernel we can indeed rigorously identify equation (1.1) as the gradient flow of the relative entropy w.r.t. the corresponding metric \mathcal{W} in the framework of the metric theory developed in [1]. So assume for the rest of this introduction that J satisfies

$$J(x+z, A+z) = J(x, A) \quad \forall x, z \in \mathbb{R}^d, A \subset \mathbb{R}^d \setminus \{x\}$$

and let m be Lebesgue measure. Then we can write $J(x, A) = \nu(A - x)$ for a Lévy measure ν on $\mathbb{R}^d \setminus \{0\}$. The operator \mathcal{L} generates a semigroup $P_t = \exp(t\mathcal{L})$ in $L^2(\mathbb{R}^d)$ that can be represented by kernel p_t :

$$P_t f(x) = \int f(y) p_t(x, dy) .$$

In fact p_t is the transition kernel of the Lévy process with characteristic triplet $(0, 0, \nu)$ in the sense of the Lévy-Khinchine formula (see e.g. [4]). In the same way \mathcal{L} generates a semigroup on $\mathcal{P}(\mathbb{R}^d)$. Under certain further regularity assumptions on the transition kernel (see Section 5 for a precise statement) we prove the following

Theorem 1.3. *The semigroup P generated by \mathcal{L} is the gradient flow of the relative entropy in the sense that it satisfies the Evolution Variational Inequality (EVI): For any $\mu \in \mathcal{P}(\mathbb{R}^d)$ and $\sigma \in \mathcal{P}_\mu$ we have*

$$\frac{1}{2} \frac{d^+}{dt} \mathcal{W}^2(P_t \mu, \sigma) + \mathcal{H}(P_t \mu) \leq \mathcal{H}(\sigma) \quad \forall t > 0 . \quad (1.8)$$

Moreover the entropy is convex along \mathcal{W} -geodesics. More precisely, let $\mu_0, \mu_1 \in \mathcal{P}(\mathbb{R}^d)$ such that $\mathcal{W}(\mu_0, \mu_1) < \infty$ and let $(\mu_t)_{t \in [0,1]}$ be a geodesic connecting μ_0 and μ_1 . Then we have

$$\mathcal{H}(\mu_t) \leq (1-t)\mathcal{H}(\mu_0) + t\mathcal{H}(\mu_1) .$$

Among several ways to characterize gradient flows in metric spaces, the EVI is one of the strongest. For example it implies geodesic convexity of the entropy (see [11]). Convexity of the entropy along \mathcal{W} -geodesics can be seen as a non-local analogue of McCann's displacement convexity [21], which corresponds to convexity along geodesics of the L^2 -Wasserstein distance. For the choice $\nu(dy) = c_\alpha |y|^{-\alpha-d} dy$ with $\alpha \in (0, 2)$ and a suitable constant c_α we obtain the following

Corollary 1.4. *The semigroup generated by the fractional Laplacian $-(-\Delta)^{\frac{\alpha}{2}}$ is the gradient flow of the relative entropy w.r.t. the metric \mathcal{W} built from the jump kernel $J_\alpha(x, dy) = c_\alpha |y - x|^{-\alpha-d} dy$.*

We expect that a similar result should also hold for semigroups associated to suitable non-homogeneous jump kernels J . It would be desirable to find examples of kernels where the entropy is strictly geodesically convex. This could be exploited to derive new functional inequalities and rates of convergence to equilibrium for the corresponding evolution equation, as has been done in the discrete setting of finite Markov chains in [14]. However, establishing a stronger $\text{EVI}(\kappa)$ in concrete examples does not seem to be an easy task and we will address this question in a forthcoming publication. Moreover, we expect that the approach presented here can be generalized in order to give a gradient flow interpretation to evolution equations associated to Lévy-type operators with both non-local and diffusion part.

Organization of the paper. In Section 2 we study the action functional \mathcal{A} and establish various properties needed in the sequel. Section 3 is devoted to an analysis of the non-local continuity equation (1.7). In Section 4 we define the metric \mathcal{W} and prove Theorem 1.2. Finally, we focus on translation invariant jump kernels and present the proof of Theorem 1.3 in Section 5.

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2. THE ACTION FUNCTIONAL

In this section we introduce and study an action functional on pairs of measures. Let us first introduce some notation. We denote by $\mathcal{P}(\mathbb{R}^d)$ the space of Borel probability measures on \mathbb{R}^d equipped with the topology of weak convergence. We let $G = \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d | x \neq y\}$ and denote by $\mathcal{M}_{loc}(G)$ the space of signed Radon measures on the open set G equipped with the weak* topology in duality with continuous functions with compact support in G .

The definition of the action functional and later the metric will depend on the choice of a function $\theta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$. We will always require it to fulfill the following assumptions:

Assumption 2.1. *The function θ has the following properties:*

- (A1) (Regularity): θ is continuous on $\mathbb{R}_+ \times \mathbb{R}_+$ and C^1 on $(0, \infty) \times (0, \infty)$;
- (A2) (Symmetry): $\theta(s, t) = \theta(t, s)$ for $s, t \geq 0$;
- (A3) (Positivity, normalisation): $\theta(s, t) > 0$ for $s, t > 0$ and $\theta(1, 1) = 1$;
- (A4) (Zero at the boundary): $\theta(0, t) = 0$ for all $t \geq 0$;
- (A5) (Monotonicity): $\theta(r, t) \leq \theta(s, t)$ for all $0 \leq r \leq s$ and $t \geq 0$;
- (A6) (Positive homogeneity): $\theta(\lambda s, \lambda t) = \lambda \theta(s, t)$ for $\lambda > 0$ and $s, t \geq 0$;
- (A7) (Concavity): the function $\theta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is concave.

It is easy to check that these assumptions imply

$$\theta(s, t) \leq \frac{s+t}{2} \quad \forall s, t \geq 0. \quad (2.1)$$

In view of applications to gradient flows of the entropy we will be mostly interested in a particular choice of θ , namely the logarithmic mean given by

$$\theta(s, t) = \int_0^1 s^\alpha t^{1-\alpha} d\alpha = \frac{s-t}{\log s - \log t}, \quad (2.2)$$

the latter expression being valid for $s, t > 0$. However, for future use we will allow for more generality in the choice of θ . Given a function $\rho : \mathbb{R}^d \rightarrow \mathbb{R}_+$ we will often write

$$\hat{\rho}(x, y) := \theta(\rho(x), \rho(y)).$$

We can now define a function $\alpha : \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+ \cup \{\infty\}$, called the action density function, by setting

$$\alpha(w, s, t) := \begin{cases} \frac{w^2}{2\theta(s, t)}, & \theta(s, t) \neq 0, \\ 0, & \theta(s, t) = 0 \text{ and } w = 0, \\ +\infty, & \theta(s, t) = 0 \text{ and } w \neq 0. \end{cases}$$

The following observation will be useful.

Lemma 2.2. *The function α is lower semicontinuous, convex and positively homogeneous, i.e.*

$$\alpha(\lambda w, \lambda s, \lambda t) = \lambda \alpha(w, s, t) \quad \forall w \in \mathbb{R}, s, t \geq 0, \lambda \geq 0.$$

Proof. This is easily checked using (A6),(A7) and the convexity of the function $(x, y) \mapsto \frac{x^2}{y}$ on $\mathbb{R} \times (0, \infty)$. \square

We will now define an action functional on pairs of measures (μ, ν) where $\mu \in \mathcal{P}(\mathbb{R}^d)$ and $\nu \in \mathcal{M}_{loc}(G)$. To μ we associate a two Radon measures in $\mathcal{M}_{loc}(G)$ by setting:

$$\mu^1(dx, dy) := J(x, dy)\mu(dx), \quad \mu^2(dx, dy) := J(y, dx)\mu(dy). \quad (2.3)$$

We can always choose a measure $\sigma \in \mathcal{M}_{loc}(G)$ such that $\mu^i = \rho^i \sigma$, $i = 1, 2$ and $\nu = w \sigma$ are all absolutely continuous with respect to σ . For example take the sum of the total variations $\sigma := |\mu^1| + |\mu^2| + |\nu|$. We can then define the *action functional* by

$$\mathcal{A}(\mu, \nu) := \int \alpha(w, \rho^1, \rho^2) d\sigma.$$

Note that this definition is independent of the choice of σ since α is positively homogeneous. Hence we can also write the action functional as

$$\mathcal{A}(\mu, \nu) = \int \alpha\left(\frac{d\lambda_1}{d|\lambda|}, \frac{d\lambda_2}{d|\lambda|}, \frac{d\lambda_3}{d|\lambda|}\right) d|\lambda|,$$

where λ is the vector valued measure given by $\lambda = (\nu, \mu^1, \mu^2)$.

In the case where the measure μ is absolutely continuous w.r.t. m the next lemma shows that the action takes a more intuitive form. For this we denote by $Jm \in \mathcal{M}_{loc}(G)$ the measure given by $Jm(dx, dy) = J(x, dy)m(dx)$.

Lemma 2.3. *Let $\mu \in \mathcal{P}(\mathbb{R}^d)$ be absolutely continuous w.r.t. m with density ρ . Further let $\nu \in \mathcal{M}_{loc}(G)$ such that $\mathcal{A}(\mu, \nu) < \infty$. Then there exist a function $w : G \rightarrow \mathbb{R}$ such that $\nu = w \hat{\rho} Jm$ and we have*

$$\mathcal{A}(\mu, \nu) = \frac{1}{2} \int |w(x, y)|^2 \hat{\rho}(x, y) J(x, dy) m(dx). \quad (2.4)$$

Proof. Choose $\lambda \in \mathcal{M}_{loc}(G)$ such that $Jm = h\lambda$ and $\nu = \tilde{w}\lambda$ are both absolutely continuous w.r.t. λ . Note that $\mu^i = \rho^i Jm$, $i = 1, 2$ with $\rho^1(x, y) = \rho(x)$ and $\rho^2(x, y) = \rho(y)$. Further, we denote by $\tilde{\rho}^i$ the density of μ^i w.r.t. λ . Now by definition,

$$\mathcal{A}(\mu, \nu) = \int \alpha(\tilde{w}, \tilde{\rho}^1, \tilde{\rho}^2) d\lambda < \infty. \quad (2.5)$$

Let $A \subset G$ such that $\int_A \theta(\rho^1, \rho^2) dJm = 0$. From the homogeneity of θ we conclude

$$0 = \int_A \theta(\rho^1, \rho^2) dJm = \int_A \theta(\tilde{\rho}^1, \tilde{\rho}^2) d\lambda,$$

i.e. $\theta(\tilde{\rho}^1, \tilde{\rho}^2) = 0$ λ -a.e. on A . Now the finiteness of the integral in (2.5) implies that $\tilde{w} = 0$ λ -a.e. on A . In other words $\nu(A) = 0$ and hence ν is absolutely continuous w.r.t. the measure $\hat{\rho} Jm$. Formula (2.4) now follows immediately from the homogeneity of α . \square

Lemma 2.4 (Lower semicontinuity of the action). *\mathcal{A} is lower semicontinuous w.r.t. weak convergence of measures. More precisely, assume that $\mu_n \rightharpoonup \mu$ weakly in $\mathcal{P}(\mathbb{R}^d)$ and $\nu_n \rightharpoonup^* \nu$ weakly* in $\mathcal{M}_{loc}(G)$. Then*

$$\mathcal{A}(\mu, \nu) \leq \liminf_n \mathcal{A}(\mu_n, \nu_n) .$$

Proof. Note that by Assumption 1.1 the weak convergence of μ_n to μ implies the weak* convergence of μ_n^i to μ^i in $\mathcal{M}_+(G)$ for $i = 1, 2$. Now the claim follows immediately from a general result on integral functionals, Proposition 2.5. \square

Proposition 2.5 ([8, Thm. 3.4.3]). *Let Ω be a locally compact Polish space and let $f : \Omega \times \mathbb{R}^n \rightarrow [0, +\infty]$ be a lower semicontinuous function such that $f(\omega, \cdot)$ is convex and positively 1-homogeneous for every $\omega \in \Omega$. Then the functional*

$$F(\lambda) = \int_{\Omega} f\left(\omega, \frac{d\lambda}{d|\lambda|}(\omega)\right) |\lambda| (d\omega)$$

is sequentially weak lower semicontinuous on the space of vector valued signed Radon measures $\mathcal{M}_{loc}(\Omega, \mathbb{R}^n)$.*

The next estimate will be crucial for establishing compactness of families of curves with bounded action in Section 3.

Lemma 2.6. i) *There exists a constant $C > 0$ such that for all $\mu \in \mathcal{P}(\mathbb{R}^d)$ and $\nu \in \mathcal{M}_{loc}(G)$ we have:*

$$\int_G (1 \wedge |x - y|) |\nu| (dx, dy) \leq C \sqrt{\mathcal{A}(\mu, \nu)} .$$

ii) *For each compact set $K \subset G$ there exists a constant $C(K) > 0$ such that for all $\mu \in \mathcal{P}(\mathbb{R}^d)$ and $\nu \in \mathcal{M}_{loc}(G)$ we have:*

$$|\nu| (K) \leq C(K) \sqrt{\mathcal{A}(\mu, \nu)} .$$

Proof. To prove i) let us define the measure $\lambda = |\mu^1| + |\mu^2| + |\nu|$ and write $\mu^i = \rho^i \lambda$, $\nu = w \lambda$. We can assume that $\mathcal{A}(\mu, \nu) < \infty$ as otherwise there is nothing to prove. This implies that the set $A = \{(x, y) \mid \alpha(w, \rho^1, \rho^2) = \infty\}$ has zero measure with respect to λ . We can now estimate:

$$\begin{aligned} & \int_G (1 \wedge |x - y|) |\nu| (dx, dy) \\ & \leq \int_G (1 \wedge |x - y|) |w| d\lambda \\ & = \int_{A^c} (1 \wedge |x - y|) \sqrt{2\theta(\rho^1, \rho^2)} \sqrt{\alpha(w, \rho^1, \rho^2)} d\lambda \\ & \leq \left(\int_G (1 \wedge |x - y|^2) 2\theta(\rho^1, \rho^2) d\lambda \right)^{\frac{1}{2}} \left(\int_G \alpha(w, \rho^1, \rho^2) d\lambda \right)^{\frac{1}{2}} \\ & \leq C \sqrt{\mathcal{A}(\mu, \nu)} . \end{aligned}$$

The last inequality follows, since by the estimate (2.1) and Assumption 1.1 we have :

$$\begin{aligned} \int_G (1 \wedge |x - y|^2) \theta(\rho^1, \rho^2) d\lambda &\leq \int_G (1 \wedge |x - y|^2) \frac{1}{2} (\rho^1 + \rho^2) d\lambda \\ &= \int_G (1 \wedge |x - y|^2) J(x, dy) \mu(dx) \\ &\leq \sup_x \int (1 \wedge |x - y|^2) J(x, dy) < \infty . \end{aligned}$$

To prove ii) we note that by a similar argument

$$|\nu|(K) \leq \left(\int_K 2J(x, dy) \mu(dx) \right)^{\frac{1}{2}} \sqrt{\mathcal{A}(\mu, \nu)} .$$

□

Lemma 2.7 (Convexity of the action). *Let $\mu^j \in \mathcal{P}(\mathbb{R}^d)$ and $\nu^j \in \mathcal{M}_{loc}(G)$ for $j = 0, 1$. For $\tau \in [0, 1]$ set $\mu^\tau = \tau\mu^1 + (1-\tau)\mu^0$ and $\nu^\tau = \tau\nu^1 + (1-\tau)\nu^0$. Then we have :*

$$\mathcal{A}(\mu^\tau, \nu^\tau) \leq \tau \mathcal{A}(\mu^1, \nu^1) + (1-\tau) \mathcal{A}(\mu^0, \nu^0) .$$

Proof. Let us fix a reference measure $\lambda \in \mathcal{M}_{loc}(G)$ such that $\mu^{j,i}, \nu^j$ for $j = 0, 1$ and $i = 1, 2$ are all absolutely continuous w.r.t. λ and write $\mu^{j,i} = \rho^{j,i} \lambda$ and $\nu^j = w^j \lambda$. Note that $\mu^{\tau,i} = \rho^{\tau,i} \lambda$ with $\rho^{\tau,i} = \tau \rho^{1,i} + (1-\tau) \rho^{0,i}$ and $\nu^\tau = w^\tau \lambda$ with $w^\tau = \tau w^1 + (1-\tau) w^0$. From the convexity of the action density function α we obtain :

$$\begin{aligned} \mathcal{A}(\mu^\tau, \nu^\tau) &= \int \alpha(w^\tau, \rho^{\tau,1}, \rho^{\tau,2}) d\lambda \\ &\leq \tau \int \alpha(w^1, \rho^{1,1}, \rho^{1,2}) d\lambda + (1-\tau) \int \alpha(w^0, \rho^{0,1}, \rho^{0,2}) d\lambda \\ &= \tau \mathcal{A}(\mu^1, \nu^1) + (1-\tau) \mathcal{A}(\mu^0, \nu^0) . \end{aligned}$$

□

We will now show that the action functional enjoys a monotonicity property under convolution if we assume that the jump kernel is translation invariant in the sense that

$$J(x - z, A - z) = J(x, A) \quad \forall x, z \in \mathbb{R}^d, A \in \mathcal{B}(\mathbb{R}^d) . \quad (2.6)$$

For the rest of this section we also assume that m is Lebesgue measure. We first need to fix a way of convoluting measure on \mathbb{R}^d and on G in a consistent manner. Let k be a convolution kernel, i.e. $k : \mathbb{R}^d \rightarrow \mathbb{R}_+$ satisfying $\int k(z) dz = 1$. Given a measure $\mu \in \mathcal{P}(\mathbb{R}^d)$, its convolution is defined as usual by

$$(\mu * k)(A) := \int k(z) \mu(A - z) dz \quad \forall A \in \mathcal{B}(\mathbb{R}^d) .$$

On the other hand given a measure $\nu \in \mathcal{M}_{loc}(G)$ we define $\nu * k \in \mathcal{M}_{loc}(G)$ by setting for all Borel measurable sets $B \subset G$

$$(\nu * k)(B) := \int k(z) \nu(B - \begin{pmatrix} z \\ z \end{pmatrix}) dz . \quad (2.7)$$

Note that this implies in particular that for every bounded function $f : G \rightarrow \mathbb{R}$ with compact support in G we have:

$$\int f(x, y) (\nu * k)(dx, dy) = \int \int k(z) f(x + z, y + z) \nu(dx, dy) dz .$$

We now have the following monotonicity property under convolution.

Proposition 2.8. *Assume that J satisfies (2.6) and let k be a convolution kernel. Then for every $\mu \in \mathcal{P}(\mathbb{R}^d)$, $\nu \in \mathcal{M}_{loc}(G)$ we have*

$$\mathcal{A}(\mu * k, \nu * k) \leq \mathcal{A}(\mu, \nu) . \quad (2.8)$$

Proof. We can assume without restriction that $\mathcal{A}(\mu, \nu)$ is finite as otherwise there is nothing to prove. Let us introduce the maps $\tau_z : x \mapsto x + z$ for $z \in \mathbb{R}^d$ and let us denote by μ_z, ν_z the push forward $(\tau_z)_* \mu = \mu(\cdot - z)$, resp. $(\tau_z \times \tau_z)_* \nu = \nu(\cdot - \begin{pmatrix} z \\ z \end{pmatrix})$. Using the convexity of the action functional, Lemma 2.7, together with its lower semicontinuity, Lemma 2.4, we see that

$$\mathcal{A}(\mu * k, \nu * k) \leq \int \mathcal{A}(\mu_z, \nu_z) k(z) dz .$$

Thus the proof is complete if we show that $\mathcal{A}(\mu_z, \nu_z) = \mathcal{A}(\mu, \nu)$ for all $z \in \mathbb{R}^d$. To this end recall the definition (2.3). Using the invariance property (2.6) it is immediate to check that $\mu_z^i = (\tau_z \times \tau_z)_* \mu^i$ for $i = 1, 2$. Now choose $\lambda \in \mathcal{M}_{loc}(G)$ with $\mu^i = \rho^i \lambda$ and $\nu = w \lambda$. Then for all $z \in \mathbb{R}^d$ we have $(\mu_z)^i = (\mu^i)_z = \rho^i(\cdot - \begin{pmatrix} z \\ z \end{pmatrix}) \lambda_z$ and $\nu_z = w(\cdot - \begin{pmatrix} z \\ z \end{pmatrix}) \lambda_z$. Hence we finally obtain

$$\begin{aligned} \mathcal{A}(\mu_z, \nu_z) &= \int \alpha \left(w(\cdot - \begin{pmatrix} z \\ z \end{pmatrix}), \rho^1(\cdot - \begin{pmatrix} z \\ z \end{pmatrix}), \rho^2(\cdot - \begin{pmatrix} z \\ z \end{pmatrix}) \right) d\lambda_z \\ &= \int \alpha(w, \rho^1, \rho^2) d\lambda = \mathcal{A}(\mu, \nu) . \end{aligned}$$

□

3. A NON-LOCAL CONTINUITY EQUATION

In this section we will consider the continuity equation

$$\partial_t \mu_t + \bar{\nabla} \cdot \nu_t = 0 \quad \text{on } (0, T) \times \mathbb{R}^d . \quad (3.1)$$

Here $(\mu_t)_{t \in [0, T]}$ and $(\nu_t)_{t \in [0, T]}$ are Borel families of measures in $\mathcal{P}(\mathbb{R}^d)$ and $\mathcal{M}_{loc}(G)$ respectively such that

$$\int_0^T \int (1 \wedge |x - y|) |\nu_t| (dx, dy) dt < \infty . \quad (3.2)$$

We suppose that (3.1) holds in the sense of distributions. More precisely, we require that for all $\varphi \in C_c^\infty((0, T) \times \mathbb{R}^d)$:

$$\int_0^T \int \partial_t \varphi_t(x) \mu_t(dx) dt + \frac{1}{2} \int_0^T \int \bar{\nabla} \varphi_t(x, y) \nu_t(dx, dy) dt = 0 . \quad (3.3)$$

Recall that for a function $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$ we denote by $\bar{\nabla}\varphi(x, y) = \varphi(y) - \varphi(x)$ the discrete gradient. Note that (3.2) is a natural integrability assumption one should make to ensure that the second term in (3.3) is well-defined. The following is an adaptation of [1, Lemma 8.1.2].

Lemma 3.1. *Let $(\mu_t)_{t \in [0, T]}$ and $(\nu_t)_{t \in [0, T]}$ be Borel families of measures in $\mathcal{P}(\mathbb{R}^d)$ and $\mathcal{M}_{loc}(G)$ satisfying (3.1) and (3.2). Then there exists a weakly continuous curve $(\tilde{\mu}_t)_{t \in [0, T]}$ such that $\tilde{\mu}_t = \mu_t$ for a.e. $t \in [0, T]$. Moreover, for every $\varphi \in C_c^\infty([0, T] \times \mathbb{R}^d)$ and all $0 \leq t_0 \leq t_1 \leq T$ we have :*

$$\int \varphi_{t_1} d\tilde{\mu}_{t_1} - \int \varphi_{t_0} d\tilde{\mu}_{t_0} = \int_{t_0}^{t_1} \int \partial_t \varphi d\mu_t dt + \frac{1}{2} \int_{t_0}^{t_1} \int \bar{\nabla} \varphi d\nu_t dt . \quad (3.4)$$

Proof. Let us set

$$V(t) := \int (1 \wedge |x - y|) |\nu_t| (dx, dy) .$$

By assumption $t \mapsto V(t)$ belongs to $L^1(0, T)$. Fix $\xi \in C_c^\infty(\mathbb{R}^d)$. We claim that the map $t \mapsto \mu_t(\xi) = \int \xi d\mu_t$ belongs to $W^{1,1}(0, T)$. Indeed, using test functions of the form $\varphi(t, x) = \eta(t)\xi(x)$ with $\eta \in C_c^\infty(0, T)$, equation (3.3) shows that the distributional derivative of $\mu_t(\xi)$ is given by

$$\dot{\mu}_t(\xi) = \frac{1}{2} \int \bar{\nabla} \xi d\nu_t$$

for a.e. $t \in (0, T)$ and we can estimate

$$|\dot{\mu}_t(\xi)| \leq \frac{1}{2} \int |\bar{\nabla} \xi| d|\nu_t| \leq \frac{1}{2} \|\xi\|_{C^1} V(t) . \quad (3.5)$$

Based on (3.5) we can argue as in [1, Lemma 8.1.2] to obtain existence of a weakly continuous representative $t \mapsto \tilde{\mu}_t$.

To prove (3.4) fix $\varphi \in C_c^\infty([0, T] \times \mathbb{R}^d)$ and choose $\eta_\varepsilon \in C_c^\infty(t_0, t_1)$ such that

$$0 \leq \eta_\varepsilon \leq 1 , \quad \lim_{\varepsilon \rightarrow 0} \eta_\varepsilon(t) = 1_{(t_0, t_1)}(t) \quad \forall t \in [0, T] , \quad \lim_{\varepsilon \rightarrow 0} \eta'_\varepsilon = \delta_{t_0} - \delta_{t_1} .$$

Now equation (3.3) implies

$$- \int_0^T \eta'_\varepsilon \int \varphi d\tilde{\mu}_t dt = \int_0^T \eta_\varepsilon \int \partial_t \varphi d\mu_t dt + \frac{1}{2} \int_0^T \eta_\varepsilon \int \bar{\nabla} \varphi d\nu_t dt .$$

Thanks to the continuity of $t \mapsto \tilde{\mu}_t$ we can pass to limit as $\varepsilon \rightarrow 0$ and obtain (3.4). \square

In view of the previous Lemma it makes sense to define solutions to the continuity equation in the following way.

Definition 3.2. *We denote by $\mathcal{CE}_T(\bar{\mu}_0, \bar{\mu}_1)$ the set of all pairs (μ, ν) satisfying the following conditions:*

$$\left\{ \begin{array}{ll} (i) & \mu : [0, T] \rightarrow \mathcal{P}(\mathbb{R}^d) \text{ is weakly continuous ;} \\ (ii) & \mu_0 = \bar{\mu}_0 , \quad \mu_T = \bar{\mu}_1 ; \\ (iii) & (\nu_t)_{t \in [0, T]} \text{ is a Borel family of measures in } \mathcal{M}_{loc}(G) ; \\ (iv) & \int_0^T \int (1 \wedge |x - y|) |\nu_t| (dx, dy) dt < \infty ; \\ (v) & \text{We have in the sense of distributions:} \\ & \partial_t \mu_t + \bar{\nabla} \cdot \nu_t = 0 . \end{array} \right. \quad (3.6)$$

The following result will allow us to extract subsequential limits from sequences of solutions to the continuity equation which have bounded action.

Proposition 3.3 (Compactness of solutions to the continuity equation). *Let (μ^n, ν^n) be a sequence in $\mathcal{CE}_T(\bar{\mu}_0, \bar{\mu}_1)$ such that*

$$\sup_n \int_0^T \mathcal{A}(\mu_t^n, \nu_t^n) dt < \infty. \quad (3.7)$$

Then there exists a couple $(\mu, \nu) \in \mathcal{CE}_T(\bar{\mu}_0, \bar{\mu}_1)$ such that up to extraction of a subsequence

$$\begin{aligned} \mu_t^n &\rightharpoonup \mu_t \quad \text{weakly in } \mathcal{P}(\mathbb{R}^d) \text{ for all } t \in [0, T], \\ \nu^n &\rightharpoonup^* \nu \quad \text{weakly}^* \text{ in } \mathcal{M}(G \times (0, T)). \end{aligned}$$

Moreover along this subsequence we have :

$$\int_0^T \mathcal{A}(\mu_t, \nu_t) dt \leq \liminf_n \int_0^T \mathcal{A}(\mu_t^n, \nu_t^n) dt.$$

Proof. For each n define the measure $\nu^n := \int_0^T \nu_t^n dt \in \mathcal{M}_{loc}(G \times (0, T))$. From Lemma 2.6 and (3.7) we infer immediately that

$$\sup_n \int_0^T \int (1 \wedge |x - y|) |\nu^n| (dx, dy) dt < \infty. \quad (3.8)$$

Moreover, for every compact set $K \subset G$ we obtain

$$\sup_n |\nu^n| (K \times [0, T]) \leq \sup_n \int_0^T |\nu_t^n| (K) dt < \infty. \quad (3.9)$$

i.e. ν^n has total variation uniformly bounded on every compact subset of $G \times [0, T]$. Hence we can extract a subsequence (still indexed by n) such that $\nu^n \rightharpoonup^* \nu$ in $\mathcal{M}_{loc}(G \times [0, T])$. By the disintegration theorem we have the representation $\nu = \int_0^T \nu_t dt$ for a Borel family (ν_t) still satisfying (3.2). Let us set $D = \{(x, x) : x \in \mathbb{R}^d\}$ and define the finite measures $\tilde{\nu}^n \in \mathcal{M}(\mathbb{R}^{2d} \times [0, T])$ given by $\tilde{\nu}^n(dx, dy) = (1 \wedge |x - y|) \nu^n(dx, dy) dt$ on $G \times [0, T]$ and $\tilde{\nu}^n(D \times [0, T]) = 0$. (3.8) implies that (up to extraction of another subsequence) $\tilde{\nu}^n \rightharpoonup^* \tilde{\nu}$ in $\mathcal{M}(\mathbb{R}^{2d} \times [0, T])$ where $\tilde{\nu}$ is defined similar to $\tilde{\nu}^n$.

Let $0 \leq t_0 \leq t_1 \leq T$ and $\xi \in C_c^\infty(\mathbb{R}^d)$. We claim that

$$\int_{t_0}^{t_1} \int \bar{\nabla} \xi d\nu_t^n dt \xrightarrow{n \rightarrow \infty} \int_{t_0}^{t_1} \int \bar{\nabla} \xi d\nu_t dt. \quad (3.10)$$

Let us define $\beta : \mathbb{R}^{2d} \times [0, T] \rightarrow \mathbb{R}$ by setting

$$\beta(x, y, t) = \begin{cases} \mathbf{1}_{(t_0, t_1)}(t) \bar{\nabla} \xi(x, y) (1 \wedge |x - y|)^{-1}, & x \neq y, \\ 0, & x = y. \end{cases}$$

Now (3.10) is equivalent to $\int \beta d\tilde{\nu}^n \rightarrow \int \beta d\tilde{\nu}$. Note that β is bounded with compact support and that the discontinuity set of β is concentrated on $\mathbb{R}^{2d} \times \{t_0, t_1\} \cup D \times [0, T]$ which is negligible for $\tilde{\nu}$. Hence the claim follows from general convergence results (see e.g. [1, Prop. 5.1.10]).

Combining now the convergence (3.10) with (3.4) for $\varphi(t, x) = \xi(x)$ and $t_0 = 0, t_1 = t$ we infer that μ_t^n converges weakly to some $\mu_t \in \mathcal{P}(\mathbb{R}^d)$

for every $t \in [0, T]$. It is easily checked that the couple (μ, ν) belongs to $\mathcal{CE}_T(\bar{\mu}_0, \bar{\mu}_1)$. As in Lemma 2.4 the lower semicontinuity now follows from Proposition 2.5 by considering $\int_0^T \mathcal{A}(\mu_t, \nu_t) dt$ as an integral functional on the space $\mathcal{M}_{loc}(G \times [0, T])$. \square

4. A NON-LOCAL TRANSPORT DISTANCE

We are now ready to give the definition of the distance \mathcal{W} . We will then establish various properties, in particular existence of geodesics. Moreover, we will characterize absolutely continuous curves in the metric space $(\mathcal{P}, \mathcal{W})$.

Definition 4.1. For $\bar{\mu}_0, \bar{\mu}_1 \in \mathcal{P}(\mathbb{R}^d)$ we define

$$\mathcal{W}(\bar{\mu}_0, \bar{\mu}_1)^2 := \inf \left\{ \int_0^1 \mathcal{A}(\mu_t, \nu_t) dt : (\mu, \nu) \in \mathcal{CE}_1(\bar{\mu}_0, \bar{\mu}_1) \right\}. \quad (4.1)$$

Let us first give an equivalent characterization of the infimum in (4.1).

Lemma 4.2. For any $T > 0$ and $\bar{\mu}_0, \bar{\mu}_1 \in \mathcal{P}(\mathbb{R}^d)$ we have :

$$\mathcal{W}(\bar{\mu}_0, \bar{\mu}_1) = \inf \left\{ \int_0^T \sqrt{\mathcal{A}(\mu_t, \nu_t)} dt : (\mu, \nu) \in \mathcal{CE}_T(\bar{\mu}_0, \bar{\mu}_1) \right\}. \quad (4.2)$$

Proof. This follows from a standard reparametrization argument. See [1, Lem. 1.1.4] or [12, Thm. 5.4] for details in similar situations. \square

The next result shows that the infimum in the definition above is in fact a minimum.

Proposition 4.3. Let $\bar{\mu}_0, \bar{\mu}_1 \in \mathcal{P}(\mathbb{R}^d)$ be such that $W := \mathcal{W}(\bar{\mu}_0, \bar{\mu}_1)$ is finite. Then the infimum in (4.1) is attained by a curve $(\mu, \nu) \in \mathcal{CE}_1(\bar{\mu}_0, \bar{\mu}_1)$ satisfying $\mathcal{A}(\mu_t, \nu_t) = W^2$ for a.e. $t \in [0, 1]$.

Proof. Existence of a minimizing curve $(\mu, \nu) \in \mathcal{CE}_1(\bar{\mu}_0, \bar{\mu}_1)$ follows immediately by the direct method taking into account Proposition 3.3. Invoking Lemma 4.2 and Jensen's inequality we see that this curve satisfies

$$\int_0^1 \sqrt{\mathcal{A}(\mu_t, \nu_t)} dt \geq W = \left(\int_0^1 \mathcal{A}(\mu_t, \nu_t) dt \right)^{\frac{1}{2}} \geq \int_0^1 \sqrt{\mathcal{A}(\mu_t, \nu_t)} dt.$$

Hence we must have $\mathcal{A}(\mu_t, \nu_t) = W^2$ for a.e. $t \in [0, T]$. \square

We now prove the first main result Theorem 1.2 announced in the introduction which we recall here for convenience.

Theorem 4.4. \mathcal{W} defines a (pseudo-) metric on $\mathcal{P}(\mathbb{R}^d)$. The topology it induces is stronger than the weak topology and bounded sets w.r.t. \mathcal{W} are weakly compact. Moreover, the map $(\mu_0, \mu_1) \mapsto \mathcal{W}(\mu_0, \mu_1)$ is lower semicontinuous w.r.t. weak convergence. For each $\tau \in \mathcal{P}(\mathbb{R}^d)$ the set $\mathcal{P}_\tau := \{\mu \in \mathcal{P}(\mathbb{R}^d) : \mathcal{W}(\mu, \tau) < \infty\}$ equipped with the distance \mathcal{W} is a complete geodesic space.

Proof. Symmetry of \mathcal{W} is obvious from the fact that $\alpha(w, \cdot, \cdot) = \alpha(-w, \cdot, \cdot)$. Equation (3.4) from Lemma 3.1 shows that two curves in \mathcal{CE}_1 can be concatenated to obtain a curve in \mathcal{CE}_2 . Hence the triangle inequality follows easily

using Lemma 4.2. To see that $\mathcal{W}(\bar{\mu}_0, \bar{\mu}_1) > 0$ whenever $\bar{\mu}_0 \neq \bar{\mu}_1$ assume that $\mathcal{W}(\bar{\mu}_0, \bar{\mu}_1) = 0$ and choose a minimizing curve $(\mu, \nu) \in \mathcal{CE}_1(\bar{\mu}_0, \bar{\mu}_1)$. Then we must have $\mathcal{A}(\mu_t, \nu_t) = 0$ and hence $\nu_t = 0$ for a.e. $t \in (0, 1)$. From the continuity equation in the form (3.4) we infer $\bar{\mu}_0 = \bar{\mu}_1$.

Let us now show that the topology induced by \mathcal{W} is stronger than the weak one. Let $\mu_n, \mu \in \mathcal{P}(\mathbb{R}^d)$ with $\mathcal{W}(\mu_n, \mu) \rightarrow 0$ and choose minimizing curves $(\mu^n, \nu^n) \in \mathcal{CE}_1(\mu_n, \mu)$. Fix a function $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$ bounded in C^1 . Using the continuity equation in the form (3.4) and Lemma 2.6 we estimate:

$$\begin{aligned} \left| \int \varphi d\mu_n - \int \varphi d\mu \right| &= \frac{1}{2} \left| \int_0^1 \int \bar{\nabla} \varphi d\nu_t^n dt \right| \\ &\leq \|\varphi\|_{C^1} \int_0^1 \int (1 \wedge |x - y|) |\nu_t^n| (dx, dy) dt \\ &\leq \|\varphi\|_{C^1} C \int_0^1 \sqrt{\mathcal{A}(\mu_t^n, \nu_t^n)} dt = \|\varphi\|_{C^1} C \cdot \mathcal{W}(\mu_n, \mu) . \end{aligned}$$

This implies $\mu_n \rightharpoonup \mu$ weakly.

The compactness assertion and lower semicontinuity of \mathcal{W} follow immediately from Proposition 3.3. Let us now fix $\tau \in \mathcal{P}(\mathbb{R}^d)$ and let $\bar{\mu}_0, \bar{\mu}_1 \in \mathcal{P}_\tau$. By the triangle inequality we have $\mathcal{W}(\bar{\mu}_0, \bar{\mu}_1) < \infty$ and hence Proposition 4.3 yields existence of minimizing curve $(\mu, \nu) \in \mathcal{CE}_1(\bar{\mu}_0, \bar{\mu}_1)$. The curve $t \mapsto \mu_t$ is then a constant speed geodesic in \mathcal{P}_τ since it satisfies

$$\mathcal{W}(\mu_s, \mu_t) = \int_s^t \sqrt{\mathcal{A}(\mu_r, \nu_r)} dr = (t - s) \mathcal{W}(\mu_0, \mu_1) \quad \forall 0 \leq s \leq t \leq 1 .$$

To show completeness let $(\mu^n)_n$ be a Cauchy sequence in \mathcal{P}_τ . In particular the sequence is bounded w.r.t. \mathcal{W} and we can find a subsequence (still indexed by n) and $\mu^\infty \in \mathcal{P}_\tau$ such that $\mu^n \rightharpoonup^* \mu^\infty$. Invoking lower semicontinuity of \mathcal{W} and the Cauchy condition we infer $\mathcal{W}(\mu^n, \mu^\infty) \rightarrow 0$ as $n \rightarrow \infty$ and $\mu^\infty \in \mathcal{P}_\tau$. \square

It is yet unclear when precisely the distance \mathcal{W} is finite. However, we will see in the next section that the distance is finite e.g. along trajectories of the semigroup associated to a translation invariant jump kernel.

The following result shows that under certain assumptions the distance \mathcal{W} can be bounded from below by the L^1 -Wasserstein distance. Recall that this distance is defined for $\mu_0, \mu_1 \in \mathcal{P}(\mathbb{R}^d)$ by

$$W_1(\mu_0, \mu_1) := \inf_{\pi} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y| \pi(dx, dy) ,$$

where the infimum is taken over all probability measures $\pi \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ whose first and second marginal are μ_0 and μ_1 respectively (see e.g. [27, Chap. 6]).

Proposition 4.5. *Assume that the jump kernel J satisfies*

$$M^2 := \sup_x \int |x - y|^2 J(x, dy) < \infty . \quad (4.3)$$

Then for any $\mu_0, \mu_1 \in \mathcal{P}(\mathbb{R}^d)$ we have the bound

$$W_1(\mu_0, \mu_1) \leq \frac{M}{\sqrt{2}} \mathcal{W}(\mu_0, \mu_1) .$$

Proof. We can assume that $\mathcal{W}(\mu_0, \mu_1) < \infty$. Take a minimizing curve $(\mu, \nu) \in \mathcal{CE}_1(\mu_0, \mu_1)$ and let $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$ be a 1-Lipschitz function. Using the continuity equation in the for (3.4) and arguing similar as in Lemma 2.6 we estimate

$$\begin{aligned} & \left| \int \varphi d\mu_n - \int \varphi d\mu \right| \\ &= \frac{1}{2} \left| \int_0^1 \int \bar{\nabla} \varphi d\nu_t^n dt \right| \\ &\leq \frac{1}{2} \int_0^1 \int |x - y| |\nu_t^n| (dx, dy) dt \\ &\leq \frac{1}{\sqrt{2}} \left(\int_0^1 \mathcal{A}(\mu_t^n, \nu_t^n) dt \right)^{\frac{1}{2}} \left(\int_0^1 \int |x - y|^2 J(x, dy) \mu_t(dx) dt \right)^{\frac{1}{2}} \\ &\leq \frac{M}{\sqrt{2}} \mathcal{W}(\mu_n, \mu) . \end{aligned}$$

Taking the supremum over all 1-Lipschitz functions φ yields the claim by Kantorovich-Rubinstein duality (see e.g. [27, 5.16]). \square

We now give a characterization of absolutely continuous curves with respect to \mathcal{W} and relate their length to their minimal action. Recall that a curve $(\mu_t)_{t \in [0, T]}$ in $\mathcal{P}(\mathbb{R}^d)$ is called absolutely continuous w.r.t. \mathcal{W} if there exists $m \in L^1(0, T)$ such that

$$\mathcal{W}(\mu_s, \mu_t) \leq \int_s^t m(r) dr \quad \forall 0 \leq s \leq t \leq T . \quad (4.4)$$

For an absolutely continuous curve the metric derivative defined by

$$|\mu'_t| := \lim_{h \rightarrow 0} \frac{\mathcal{W}(\mu_{t+h}, \mu_t)}{|h|}$$

exists for a.e. $t \in [0, T]$ and is the minimal m in (4.4).

Proposition 4.6 (Metric velocity). *A curve $(\mu_t)_{t \in [0, T]}$ is absolutely continuous with respect to \mathcal{W} if and only if there exists a Borel family $(\nu_t)_{t \in [0, T]}$ such that $(\mu, \nu) \in \mathcal{CE}_T$ and*

$$\int_0^T \sqrt{\mathcal{A}(\mu_t, \nu_t)} dt < \infty .$$

In this case we have $|\mu'_t|^2 \leq \mathcal{A}(\mu_t, \nu_t)$ for a.e. $t \in [0, T]$. Moreover, there exists a unique Borel family $\tilde{\nu}_t$ with $(\mu, \tilde{\nu}) \in \mathcal{CE}_T$ such that

$$|\mu'_t|^2 = \mathcal{A}(\mu_t, \tilde{\nu}_t) \quad \text{for a.e. } t \in [0, T] . \quad (4.5)$$

Proof. The proof follows from the very same arguments as in [12, Thm. 5.17]. \square

We can describe the optimal velocity measures $\tilde{\nu}_t$ appearing in the preceding proposition in more detail. We define

$$T_\mu \mathcal{P}(\mathbb{R}^d) := \left\{ \nu \in \mathcal{M}_{loc}(G) : \mathcal{A}(\mu, \nu) < \infty, \right. \\ \left. \mathcal{A}(\mu, \nu) \leq \mathcal{A}(\mu, \nu + \eta) \ \forall \eta : \bar{\nabla} \cdot \eta = 0 \right\}. \quad (4.6)$$

Here $\bar{\nabla} \cdot \eta = 0$ is understood in a weak sense, i.e.

$$\frac{1}{2} \int \bar{\nabla} \xi(x, y) \eta(dx, dy) = 0 \quad \forall \xi \in C_c^\infty(\mathbb{R}^d).$$

Corollary 4.7. *Let $(\mu, \nu) \in \mathcal{CE}_T$ such that the curve $t \mapsto \mu_t$ is absolutely continuous w.r.t. \mathcal{W} . Then ν satisfies (4.5) if and only if $\nu_t \in T_{\mu_t} \mathcal{P}(\mathbb{R}^d)$ for a.e. $t \in [0, T]$.*

In the light of the formal Riemannian interpretation of the distance \mathcal{W} we view $T_\mu \mathcal{P}(\mathbb{R}^d)$ as the tangent space to $\mathcal{P}(\mathbb{R}^d)$ at the measure μ . If μ is absolutely continuous with respect to m we can give an explicit description of $T_\mu \mathcal{P}(\mathbb{R}^d)$ as a subspace of an L^2 space. For this recall that we denote by $Jm \in \mathcal{M}_{loc}(G)$ the measure given by $Jm(dx, dy) = J(x, dy)m(dx)$.

Proposition 4.8. *Let $\mu = \rho m \in \mathcal{P}(\mathbb{R}^d)$. Then we have $\nu \in T_\mu \mathcal{P}(\mathbb{R}^d)$ if and only if $\nu = w \hat{\rho} Jm$ is absolutely continuous w.r.t. the measure $\hat{\rho} Jm$ and*

$$w \in \overline{\{\bar{\nabla} \varphi \mid \varphi \in C_c^\infty(\mathbb{R}^d)\}}^{L^2(\hat{\rho} Jm)} =: T_\rho.$$

Proof. If $\mathcal{A}(\mu, \nu)$ is finite we infer from Lemma 2.3 that $\nu = w \hat{\rho} Jm$ for some density $w : G \rightarrow \mathbb{R}$ and that $\mathcal{A}(\mu, \nu) = \|w\|_{L^2(\hat{\rho} Jm)}^2$. Now the optimality condition in (4.6) is equivalent to

$$\|w\|_{L^2(\hat{\rho} Jm)} \leq \|w + v\|_{L^2(\hat{\rho} Jm)} \quad \forall v \in N_\rho,$$

where $N_\rho := \{v \in L^2(\hat{\rho} Jm) : \int \bar{\nabla} \xi v \hat{\rho} dJm = 0 \ \forall \xi \in C_c^\infty(\mathbb{R}^d)\}$. This implies the assertion of the proposition after noting that N_ρ is the orthogonal complement in L^2 of T_ρ . \square

The convexity and monotonicity properties of the action functional established in Section 2 extend naturally to the distance function.

Proposition 4.9 (Convexity of the distance). *Let $\mu_0^j, \mu_1^j \in \mathcal{P}(\mathbb{R}^d)$ for $j = 0, 1$. For $\tau \in [0, 1]$ and $k = 0, 1$ set $\mu_k^\tau = \tau \mu_k^1 + (1 - \tau) \mu_k^0$. Then we have :*

$$\mathcal{W}(\mu_0^\tau, \mu_1^\tau)^2 \leq \tau \mathcal{W}(\mu_0^1, \mu_1^1)^2 + (1 - \tau) \mathcal{W}(\mu_0^0, \mu_1^0)^2.$$

Proof. We can assume that $\mathcal{W}(\mu_0^j, \mu_1^j)$ is finite and choose minimizing curves $(\mu^j, \nu^j) \in \mathcal{CE}_1(\mu_0^j, \mu_1^j)$. Then for $t \in [0, 1]$ set $\mu_t^\tau = \tau \mu_t^1 + (1 - \tau) \mu_t^0$ and $\nu_t^\tau = \tau \nu_t^1 + (1 - \tau) \nu_t^0$. Observe that $(\mu^\tau, \nu^\tau)_t \in \mathcal{CE}_1(\mu_0^\tau, \mu_1^\tau)$. From the definition of \mathcal{W} and the convexity of \mathcal{A} as stated in Lemma 2.7 we infer

$$\mathcal{W}(\mu_0^\tau, \mu_1^\tau)^2 \leq \int_0^1 \mathcal{A}(\mu_t^\tau, \nu_t^\tau) dt \leq \int_0^1 \tau \mathcal{A}(\mu_t^1, \nu_t^1) + (1 - \tau) \mathcal{A}(\mu_t^0, \nu_t^0) dt \\ = \tau \mathcal{W}(\mu_0^1, \mu_1^1)^2 + (1 - \tau) \mathcal{W}(\mu_0^0, \mu_1^0)^2.$$

\square

Proposition 4.10 (Monotonicity under convolution). *Let $\mu_0, \mu_1 \in \mathcal{P}(\mathbb{R}^d)$. Assume that J satisfies (2.6) and let m be Lebesgue measure. Let k be a convolution kernel. Then we have*

$$\mathcal{W}(\mu_0 * k, \mu_1 * k) \leq \mathcal{W}(\mu_0, \mu_1) .$$

If we set $k_\varepsilon(x) = \varepsilon^{-d}k(x/\varepsilon)$, then as $\varepsilon \searrow 0$ we have

$$\mathcal{W}(\mu_0 * k_\varepsilon, \mu_1 * k_\varepsilon) \longrightarrow \mathcal{W}(\mu_0, \mu_1) .$$

Proof. Assume that $\mathcal{W}(\mu_0, \mu_1)$ is finite, as otherwise there is nothing to prove. Let $(\mu, \nu) \in \mathcal{CE}_1(\mu_0, \mu_1)$ be a minimizing curve according to Proposition 4.3. Define $\tilde{\mu}_t = \mu_t * k, \tilde{\nu}_t = \nu_t * k$. We claim that $(\tilde{\mu}, \tilde{\nu}) \in \mathcal{CE}_1(\mu_0 * k, \mu_1 * k)$. Indeed, let us show that the continuity equation (v) in (3.6) holds for $(\tilde{\mu}, \tilde{\nu})$. The other properties are equally easy to verify. So let $\varphi \in C_c^\infty((0, 1) \times \mathbb{R}^d)$ and set $\tilde{\varphi}(t, x) = \int \varphi(t, x + z)k(z)dz$. Using the continuity equation for (μ, ν) and (2.7) we obtain

$$\begin{aligned} \int \partial_t \varphi d\tilde{\mu}_t dt &= \int \partial_t \varphi(t, x + z)k(z)dz \mu_t(dx) dt \\ &= \int \partial_t \tilde{\varphi} d\mu_t dt = -\frac{1}{2} \int \bar{\nabla} \tilde{\varphi} d\nu_t dt \\ &= -\frac{1}{2} \int \bar{\nabla} \varphi(t, x + z, y + z)k(z)\nu_t(dx, dy) dz dt \\ &= -\frac{1}{2} \int \bar{\nabla} \varphi d\tilde{\nu}_t dt . \end{aligned}$$

Now the first assertion follows immediately from Proposition 2.8. This in turn together with weak lower semicontinuity of \mathcal{W} (see Theorem 4.4) yields the second assertion. \square

5. GEODESIC CONVEXITY AND GRADIENT FLOW OF THE ENTROPY

In this section we focus on a translation invariant jump kernel J and will identify the evolution equation (1.1) as the gradient flow of the relative entropy in the framework of gradient flows in metric spaces developed in [1]. So let us assume from now on that J satisfies

$$J(x - z, A) = J(x, A + z) \quad \forall x, z \in \mathbb{R}^d, A \in \mathcal{B}(\mathbb{R}^d)$$

and that m is Lebesgue measure on \mathbb{R}^d . Moreover we assume that θ is the logarithmic mean defined by (2.2). Under this assumptions we can write

$$J(x, A) = \nu(A - x) \quad \forall x \in \mathbb{R}^d, A \in \mathcal{B}(\mathbb{R}^d) ,$$

where ν is a Lévy measure, i.e. a Borel measure on $\mathbb{R}^d \setminus \{0\}$ satisfying

$$\int (1 \wedge |y|^2) \nu(dy) < \infty .$$

Now the evolution equation takes the form

$$\partial_t \rho = \mathcal{L} \rho ,$$

where the operator \mathcal{L} is given by

$$\mathcal{L} \rho(x) := \int (\rho(x + y) - \rho(x) - y \cdot \nabla \rho(x) \mathbf{1}_{\{|y| \leq 1\}}) \nu(dy) .$$

Note that \mathcal{L} is also the generator of the Lévy process X with vanishing drift and diffusion and with Lévy measure ν (see e.g. [4] for background on Lévy processes). It is a pseudo differential operator whose symbol is given by the Lévy-Khinchine formula

$$\eta(\xi) = \int e^{i\langle y, \xi \rangle} - 1 - i\langle y, \xi \rangle \mathbf{1}_{\{|y| \leq 1\}} \nu(dy) .$$

This means that $\mathcal{F}(\mathcal{L}\rho) = \eta\mathcal{F}(\rho)$, where \mathcal{F} denotes the Fourier transform. Recall that the law of X_t can be given explicitly in terms of its Fourier transformation. Namely, we have

$$\mathbb{E}[\exp(i\langle \xi, X_t \rangle)] = \exp(t\eta(\xi)) .$$

Throughout this section we will make the following assumption on ν in terms of the law of the associated Lévy process.

Assumption 5.1. *Assume that the law of the process X_t has a density ψ_t such that $\psi_t > 0$ for all $t > 0$. Moreover, assume that $\psi : (0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}_+$ is such that $\psi_t, \mathcal{L}\psi_t$ are rapidly decreasing functions locally uniformly in t .*

Remark 5.2. This is a technical assumption made to simplified the presentation. It is used to ensure convergence of integrals in the proof of Theorem 5.5 and could be weakened substantially. Still, Assumption 5.1 is fulfilled for example, when $\nu(dy) = c_\alpha |y|^{-\alpha-d}$ for $\alpha \in (0, 2)$. For a suitable constant c_α the Lévy process X is then the symmetric, isotropic α -stable process and the symbol is given by $\eta(\xi) = |\xi|^\alpha$.

Recall that a smooth function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is called rapidly decreasing if $|x^\beta D^\alpha f(x)| \rightarrow 0$ as $|x| \rightarrow \infty$ for any multi-indices α, β . We obtain a semigroup $(P_t)_{t \geq 0}$ on $\mathcal{P}(\mathbb{R}^d)$ endowed with the distance \mathcal{W} by setting

$$P_t[\mu] := \mu * \psi_t .$$

For $\nu \in \mathcal{M}(G)$ we set

$$P_t[\nu] := \nu * \psi_t ,$$

with the convolution being understood in the sense of (2.7). Proposition 4.10 shows that P is a C^0 -semigroup in the sense that $P_t[\mu] \rightarrow \mu$ weakly as $t \rightarrow 0$. Moreover, $P_t[\mu] = \rho_t m$ is absolutely continuous w.r.t. Lebesgue measure for any $\mu \in \mathcal{P}(\mathbb{R}^d)$ and the density ρ_t satisfies $\partial_t \rho_t = \mathcal{L}\rho_t$.

The notion of gradient flow can be defined in abstract metric spaces and has been studied extensively in this setting (see [1]). Of particular interest are gradient flows of functionals that are geodesically (semi-) convex. In this situation the gradient flow is characterized by the so called “Evolution Variational Inequality” (EVI). We adopt the following definition.

Definition 5.3. *Let (X, d) be a metric space and $F : X \rightarrow (-\infty, \infty]$ a lower semicontinuous function. Further let $(S_t)_{t \geq 0}$ be a C^0 -semigroup on X and $\lambda \in \mathbb{R}$. S is called the (λ) -gradient flow of F if $S_t(X) \subset D(F)$ for all $t > 0$, the map $t \mapsto F(S_t(u))$ is non-increasing for all $u \in X$ and if for all $u \in X, v \in D(F), t \geq 0$:*

$$\frac{1}{2} \frac{d^+}{dt} d^2(S_t(u), v) + \frac{\lambda}{2} d^2(S_t(u), v) + F(S_t(u)) \leq F(v) . \quad (5.1)$$

Here $D(F) := \{x \in X \mid F(x) < \infty\}$ denotes the proper domain of the function F .

We will apply this definition in the case where $X = \mathcal{P}(\mathbb{R}^d)$ and F is the relative entropy \mathcal{H} defined for $\mu \in \mathcal{P}(\mathbb{R}^d)$ by

$$\mathcal{H}(\mu) := \begin{cases} \int \rho \log \rho \, dm, & \text{if } \mu = \rho dm \text{ and } \int (\rho \log \rho)_+ dm < \infty \\ +\infty, & \text{else.} \end{cases}$$

Let us start by stating a result giving the entropy production along the semigroup P . As before, we will denote by $Jm \in \mathcal{M}_{loc}(G)$ the measure given by $Jm(dx, dy) = J(x, dy)m(dx)$. For a probability measure $\mu \in \mathcal{P}(\mathbb{R}^d)$ we define a non-local analogue of the Fisher information by

$$\mathcal{I}(\mu) := \begin{cases} \frac{1}{2} \int \bar{\nabla} \rho \bar{\nabla} \log \rho \, d(Jm), & \text{if } \mu = \rho m \text{ and } \rho > 0, \\ +\infty, & \text{else.} \end{cases} \quad (5.2)$$

Proposition 5.4. *Let $\mu \in \mathcal{P}(\mathbb{R}^d)$ and set $\mu_t = \rho_t m := P_t[\mu]$. For every $t > 0$ we have $\mathcal{H}(\mu_t) \in (-\infty, \infty)$ and $\mathcal{I}(\mu_t) < \infty$. Moreover, we have the energy identity*

$$\mathcal{H}(\mu_t) - \mathcal{H}(\mu_s) = - \int_s^t \mathcal{I}(\mu_r) \, dr \quad \forall t \geq s > 0. \quad (5.3)$$

In particular the map $t \mapsto \mathcal{H}(\mu_t)$ is non-increasing.

Proof. Finiteness of $\mathcal{H}(\mu_t)$ follows readily from the fact that ψ_t is rapidly decreasing. We prove (5.3) by approximating \mathcal{H} with functionals \mathcal{H}_n . Let us set

$$f_n(u) := \int_0^u \max(1 + \log(r), -n) \, dr. \quad (5.4)$$

Then we have $f_n(u) \searrow u \log(u)$ and $f'_n(u) \searrow 1 + \log(u)$ as $n \rightarrow \infty$. For $\mu = \rho m \in \mathcal{P}(\mathbb{R}^d)$ we set $\mathcal{H}_n(\mu) := \int f_n(\rho) dm$. Now we calculate

$$\begin{aligned} \mathcal{H}_n(\mu_t) - \mathcal{H}_n(\mu_s) &= \int f_n(\rho_t) - f_n(\rho_s) \, dm \\ &= \int \int_s^t f'_n(\rho_r) \partial_r \rho_r \, dr dm = \int \int_s^t f'_n(\rho_r) \mathcal{L} \rho_r \, dr dm \\ &= \frac{1}{2} \int_s^t \int \bar{\nabla} f'_n(\rho_r) \bar{\nabla} \rho_r \, d(Jm) dr. \end{aligned}$$

The interchange of integrals and integration by parts are easily justified by the fact that $f'_n(\rho_r)$ is bounded and $\mathcal{L} \rho_r$ is rapidly decreasing locally uniformly in r . Letting finally $n \rightarrow \infty$ we obtain (5.3) by monotone convergence of both the left and right hand sides. \square

We will now show that the semigroup (P_t) is the gradient flow of the relative entropy with respect to the distance \mathcal{W} . Our strategy of proof is inspired by an argument developed in [11] and used in a similar form in [12, Thm. 5.29]. Recall that \mathcal{W} is a pseudo distance, thus it is necessary to consider the sets $\mathcal{P}_\tau := \{\mu \in \mathcal{P}(\mathbb{R}^d) : \mathcal{W}(\mu, \tau) < \infty\}$ for a given $\tau \in \mathcal{P}(\mathbb{R}^d)$. The following two results are a restatement of Theorem 1.3.

Theorem 5.5. *Let $\mu \in \mathcal{P}(\mathbb{R}^d)$ and set $\mu_t := P_t[\mu]$. Then $\mu_t \in D(\mathcal{H}) \cap \mathcal{P}_\mu$ for all $t > 0$ and the map $t \mapsto \mathcal{H}(\mu_t)$ is non-increasing. Moreover, for any $\sigma \in \mathcal{P}_\mu$ the Evolution Variational Inequality holds:*

$$\frac{1}{2} \frac{d^+}{dt} \mathcal{W}^2(\mu_t, \sigma) + \mathcal{H}(\mu_t) \leq \mathcal{H}(\sigma) \quad \forall t > 0. \quad (5.5)$$

Proof. The first statement is a direct consequence of Proposition 5.4. For the second statement it is sufficient to assume $\mu \in D(\mathcal{H})$ and prove the inequality at $t = 0$. So let $\sigma \in D(\mathcal{H})$ and let $(\mu_s, \nu_s)_{s \in [0,1]}$ be a minimizing curve $\mu_0 := \sigma$ to $\mu_1 := \mu$. We set

$$\begin{aligned} \mu_{s,t}^\varepsilon &= \rho_{s,t}^\varepsilon m := P_{st+\varepsilon}[\mu_s] \quad \text{and} \\ \tilde{\nu}_{s,t}^\varepsilon &= \tilde{v}_{s,t}^\varepsilon Jm := P_{st+\varepsilon}[\nu_s]. \end{aligned}$$

The couple $(\mu_{s,t}^\varepsilon, \tilde{\nu}_{s,t}^\varepsilon)$ does not satisfy the continuity equation. Hence we make the correction

$$\nu_{s,t}^\varepsilon = v_{s,t}^\varepsilon Jm := (\tilde{v}_{s,t}^\varepsilon - t \bar{\nabla} \rho_{s,t}^\varepsilon) Jm.$$

We will need the following result whose proof we postpone for the moment.

Claim 5.6. We have $(\mu_{s,t}^\varepsilon, \nu_{s,t}^\varepsilon) \in \mathcal{CE}_1(\sigma_\varepsilon, \mu_{\varepsilon+t})$ and moreover,

$$\mathcal{H}(\mu_{\varepsilon+t}) - \mathcal{H}(\mu_\varepsilon) = -\frac{1}{2} \int_0^1 \int \bar{\nabla} \log \rho_{s,t}^\varepsilon v_{s,t}^\varepsilon d(Jm) ds. \quad (5.6)$$

From the definition of the distance \mathcal{W} we now obtain the estimate

$$\mathcal{W}(\mu_{t+\varepsilon}, \sigma_\varepsilon)^2 \leq \int_0^1 \mathcal{A}(\mu_{s,t}^\varepsilon, \nu_{s,t}^\varepsilon) ds. \quad (5.7)$$

Recall the notation $\hat{\rho}(x, y) = \theta(\rho(x), \rho(y))$ with θ being the logarithmic mean here. We can further estimate

$$\begin{aligned} \mathcal{A}(\mu_{s,t}^\varepsilon, \nu_{s,t}^\varepsilon) &= \int \frac{|v_{s,t}^\varepsilon|^2}{2\hat{\rho}_{s,t}^\varepsilon} d(Jm) \\ &= \int (|\tilde{v}_{s,t}^\varepsilon|^2 - 2t \bar{\nabla} \rho_{s,t}^\varepsilon v_{s,t}^\varepsilon - t^2 |\bar{\nabla} \rho_{s,t}^\varepsilon|^2) \frac{1}{2\hat{\rho}_{s,t}^\varepsilon} d(Jm) \\ &\leq \mathcal{A}(\mu_{s,t}^\varepsilon, \tilde{\nu}_{s,t}^\varepsilon) - t \int \bar{\nabla} \log \rho_{s,t}^\varepsilon v_{s,t}^\varepsilon d(Jm) \\ &\leq \mathcal{A}(\mu_s, \nu_s) - t \int \bar{\nabla} \log \rho_{s,t}^\varepsilon v_{s,t}^\varepsilon d(Jm), \end{aligned}$$

where we have dropped the quadratic term in t and used the monotonicity under convolution (Proposition 2.8) in the last inequality. Integration over s from 0 to 1 and using (5.6) gives

$$\frac{1}{2} \mathcal{W}(\mu_{t+\varepsilon}, \sigma_\varepsilon)^2 \leq \frac{1}{2} \mathcal{W}(\mu, \sigma)^2 - t \cdot (\mathcal{H}(\mu_{t+\varepsilon}) - \mathcal{H}(\sigma_\varepsilon)).$$

By lower semicontinuity of \mathcal{W} (see Theorem 4.4) and continuity of \mathcal{H} along the semigroup we can take the limit $\varepsilon \rightarrow 0$ and obtain

$$\frac{1}{2} \mathcal{W}(\mu_t, \sigma)^2 \leq \frac{1}{2} \mathcal{W}(\mu, \sigma)^2 - t \cdot (\mathcal{H}(\mu_t) - \mathcal{H}(\sigma)).$$

Finally, rearranging terms and letting $t \searrow 0$ yields (5.5).

Proof of Claim 5.6. For the proof we first need two estimates. First note that

$$\int_0^1 \mathcal{I}(\mu_{s,t}^\varepsilon) \, ds < \infty . \quad (5.8)$$

Indeed, by convexity of the map $(u, v) \mapsto (u - v)(\log u - \log v)$ we have that $\mathcal{I}(\mu * \psi_t) \leq \mathcal{I}(\psi_t m)$ for every $\mu \in \mathcal{P}(\mathbb{R}^d)$. Hence we conclude from Proposition 5.4 that

$$\int_0^1 \mathcal{I}(\mu_{s,t}^\varepsilon) \, ds \leq \int_0^1 \mathcal{I}(\psi_{\varepsilon+st} m) \, ds = \mathcal{H}(\psi_\varepsilon m) - \mathcal{H}(\psi_{\varepsilon+t} m) < \infty .$$

From this we conclude that the curve $(\mu_{\cdot,t}^\varepsilon, \nu_{\cdot,t}^\varepsilon)$ has finite action. Indeed,

$$\begin{aligned} A &:= \int_0^1 \int \frac{|v_{s,t}^\varepsilon|^2}{2\hat{\rho}_{s,t}^\varepsilon} \, d(Jm) \, ds \\ &\leq \int_0^1 \int 2 \frac{|\tilde{v}_{s,t}^\varepsilon|^2}{2\hat{\rho}_{s,t}^\varepsilon} + 2t^2 \frac{|\bar{\nabla} \rho_{s,t}^\varepsilon|^2}{2\hat{\rho}_{s,t}^\varepsilon} \, d(Jm) \, ds \\ &\leq 2 \int_0^1 \mathcal{A}(\mu_s, \nu_s) \, ds + 2t^2 \int_0^1 \mathcal{I}(\mu_{s,t}^\varepsilon) \, ds < \infty , \end{aligned}$$

where we use Proposition 2.8 in the last inequality. Using Lemma 2.6 and the previous estimate we see that $\nu_{\cdot,t}^\varepsilon$ satisfies the integrability condition (iv) in Definition 3.2. The other conditions are also easily checked. Hence we see $(\mu_{\cdot,t}^\varepsilon, \nu_{\cdot,t}^\varepsilon) \in \mathcal{CE}_1(\sigma_\varepsilon, \mu_{\varepsilon+t})$.

Now let us prove (5.6). By a simple convolution argument we can assume that $\rho_{s,t}^\varepsilon$ is differentiable in s . Let f_n be the function defined by (5.4) and set $f(u) = u \log(u)$ for $u \geq 0$. Now we calculate

$$\mathcal{H}_n(\mu_{\varepsilon+t}) - \mathcal{H}_n(\mu_\varepsilon) = \int \int_0^1 f'_n(\rho_{s,t}^\varepsilon) \partial_s \rho_{s,t}^\varepsilon \, ds \, dm .$$

Note that the map $x \mapsto f'_n(\rho_{s,t}^\varepsilon(x))$ is bounded and Lipschitz uniformly in $s \in [0, 1]$. Using the integrability condition (iv) from Definition 3.2 we can approximate it by functions in $C_c^\infty((0, 1) \times \mathbb{R}^d)$ and obtain by the continuity equation

$$\mathcal{H}_n(\mu_{\varepsilon+t}) - \mathcal{H}_n(\mu_\varepsilon) = -\frac{1}{2} \int_0^1 \int \bar{\nabla} f'_n(\rho_{s,t}^\varepsilon) v_{s,t}^\varepsilon \, d(Jm) \, ds . \quad (5.9)$$

By monotone convergence the left hand side of (5.9) converges to the left hand side of (5.6). It remains to prove convergence of the right hand side. Using Hölder inequality we estimate

$$\begin{aligned} &\left| \int_0^1 \int \bar{\nabla} (f'(\rho_{s,t}^\varepsilon) - f'_n(\rho_{s,t}^\varepsilon)) \, d\nu_{s,t}^\varepsilon \, ds \right| \\ &\leq \int_0^1 \int |\bar{\nabla} (f'(\rho_{s,t}^\varepsilon) - f'_n(\rho_{s,t}^\varepsilon))| |w_{s,t}^\varepsilon| \, d(Jm) \, ds \\ &\leq A^{\frac{1}{2}} \left(\int_0^1 \int |\bar{\nabla} (f'(\rho_{s,t}^\varepsilon) - f'_n(\rho_{s,t}^\varepsilon))|^2 2\hat{\rho}_{s,t}^\varepsilon \, d(Jm) \, ds \right)^{\frac{1}{2}} . \end{aligned}$$

The integrand in the last term is bounded as

$$|\bar{\nabla}(f'(\rho_{s,t}^\varepsilon) - f'_n(\rho_{s,t}^\varepsilon))|^2 \hat{\rho}_{s,t}^\varepsilon \leq |\bar{\nabla}f'(\rho_{s,t}^\varepsilon)|^2 \hat{\rho}_{s,t}^\varepsilon = \bar{\nabla} \log \rho_{s,t}^\varepsilon \bar{\nabla} \rho_{s,t}^\varepsilon .$$

With the help of (5.8) and dominated convergence we conclude convergence of the right hand side of (5.9) to the right hand side of (5.6). \square

\square

Corollary 5.7. *The entropy is convex along \mathcal{W} -geodesics. More precisely, let $\mu_0, \mu_1 \in \mathcal{P}(\mathbb{R}^d)$ such that $\mathcal{W}(\mu_0, \mu_1) < \infty$ and let $(\mu_t)_{t \in [0,1]}$ be a geodesic connecting μ_0 and μ_1 . Then we have*

$$\mathcal{H}(\mu_t) \leq (1-t)\mathcal{H}(\mu_0) + t\mathcal{H}(\mu_1) .$$

Proof. This is a direct consequence of Theorem 5.5 and the fact, proved in [11, Thm. 3.2], that in a general setting the Evolution Variational Inequality implies geodesic convexity. \square

We finish by giving an equivalent and more intuitive definition of the distance \mathcal{W} in the present setting of a translation invariant jump kernel J . We show that it coincides with $\widetilde{\mathcal{W}}$ defined in (1.5). We introduce the following shorthand notation. Given functions $\rho : \mathbb{R}^d \rightarrow \mathbb{R}_+$ and $\psi : \mathbb{R}^d \rightarrow \mathbb{R}$ we write

$$\mathcal{A}'(\rho, \psi) := \frac{1}{2} \int (\psi(y) - \psi(x))^2 \hat{\rho}(x, y) J(x, dy) m(dx) .$$

For two probability densities $\bar{\rho}_0, \bar{\rho}_1$ w.r.t. m and $T > 0$ let us denote by $\mathcal{CE}'_T(\bar{\rho}_0, \bar{\rho}_1)$ the collection of pairs (ρ, ψ) satisfying the following conditions:

$$\left\{ \begin{array}{ll} (i) & \rho : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}_+ \text{ is measurable ;} \\ (ii) & \rho_t \text{ is a probability density for all } t \in [0, T] ; \\ (iii) & \text{The curve } t \mapsto \mu_t := \rho_t m \text{ is weakly continuous ;} \\ (iv) & \psi : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R} \text{ is measurable ;} \\ (v) & \partial_t \rho_t + \bar{\nabla} \cdot (\hat{\rho}_t \bar{\nabla} \psi_t) = 0 , \rho_0 = \bar{\rho}_0 , \rho_T = \bar{\rho}_1 . \end{array} \right. \quad (5.10)$$

Here the continuity equation (v) is understood in the sense that for every test function $\varphi \in C_c^\infty((0, T) \times \mathbb{R}^d)$ we have

$$\int_0^1 \int \partial_t \varphi \rho_t dm dt + \frac{1}{2} \int_0^1 \int \bar{\nabla} \varphi(x, y) \bar{\nabla} \psi_t(x, y) \hat{\rho}_t(x, y) J(x, dy) m(dy) dt = 0 .$$

Proposition 5.8. *Assume that m is Lebesgue measure and that $J(x, dy) = j(y-x)dy$ for a function $j : \mathbb{R}^d \setminus \{0\} \rightarrow \mathbb{R}^+$ that is strictly positive. Moreover, assume that J satisfies 5.1. Let $\bar{\mu}_i = \bar{\rho}_i m \in \mathcal{P}(\mathbb{R}^d)$ for $i = 0, 1$ such that $\mathcal{I}(\bar{\mu}_i)$ is finite. Then we have*

$$\mathcal{W}(\bar{\mu}_0, \bar{\mu}_1)^2 = \inf \left\{ \int_0^1 \mathcal{A}'(\rho_t, \psi_t) dt : (\rho, \psi) \in \mathcal{CE}'_1(\bar{\rho}_0, \bar{\rho}_1) \right\} .$$

Note that the assumptions above on the jump kernel J are satisfied by the kernel J_α associated to the fractional Laplacian.

Proof. The inequality ' \leq ' follows easily by noting that the infimum in the definition of \mathcal{W} is taken over a larger set. Indeed, given a pair $(\rho, \psi) \in \mathcal{CE}'_1(\bar{\rho}_0, \bar{\rho}_1)$ such that $\int_0^1 \mathcal{A}'(\rho_t, \psi_t) dt$ is finite we set $\mu_t = \rho_t m$ and define $\nu_t \in \mathcal{M}_{loc}(G)$ by $\nu_t(dx, dy) = \bar{\nabla} \psi_t(x, y) \hat{\rho}_t(x, y) J(x, dy) m(dx)$. Then obviously

we have $\mathcal{A}'(\rho_t, \psi_t) = \mathcal{A}(\mu_t, \nu_t)$ and it is easily checked using Lemma 2.6 that $(\mu, \nu) \in \mathcal{CE}_1(\bar{\mu}_0, \bar{\mu}_1)$.

Let us now prove the opposite inequality ' \geq '. To this end, note that by a reparametrization argument similar to Lemma 4.2 the square root of the infimum on the right hand side coincides with

$$\inf \left\{ \int_0^T \sqrt{\mathcal{A}'(\rho_t, \psi_t)} dt : (\rho, \psi) \in \mathcal{CE}'_T(\bar{\rho}_0, \bar{\rho}_1) \right\}.$$

We set $\mu_t^{i,\varepsilon} := P_t[\bar{\mu}_i] = \rho_t^{i,\varepsilon} m$ and $\psi_t^{i,\varepsilon} = \log \rho_t^{i,\varepsilon}$ for $i = 0, 1$ and $t \in (0, \varepsilon]$. It is easily checked, that the pair $(\rho^{i,\varepsilon}, \psi^{i,\varepsilon})$ belongs to $\mathcal{CE}'_\varepsilon(\bar{\rho}_i, \rho_1^{i,\varepsilon})$. Using the monotonicity of \mathcal{I} under convolution as in the proof of Claim 5.6 we infer that

$$L^{i,\varepsilon} := \int_0^\varepsilon \sqrt{\mathcal{A}'(\rho_t^{i,\varepsilon}, \psi_t^{i,\varepsilon})} dt = \int_0^\varepsilon \sqrt{\mathcal{I}(\mu_t^{i,\varepsilon})} dt \leq \varepsilon \sqrt{\mathcal{I}(\bar{\mu}_i)}.$$

Now let $(\mu, \nu) \in \mathcal{CE}_1(\bar{\mu}_0, \bar{\mu}_1)$ be a geodesic and set $\mu_t^\varepsilon := P_\varepsilon[\mu_t] = \rho_t^\varepsilon m$. Proposition 4.6 and the proof of Proposition 4.10 show that the curve $t \mapsto \mu_t^\varepsilon$ is absolutely continuous w.r.t. \mathcal{W} and thus there is a family of optimal velocity measures $\tilde{\nu}^\varepsilon$. By Proposition 4.8 we have that $\tilde{\nu}_t^\varepsilon = w_t^\varepsilon \hat{\rho}_t^\varepsilon Jm$ where w_t^ε belongs to T_ρ^ε . Note that $\rho_t^\varepsilon > 0$ by Assumption 5.1 and thus $\hat{\rho}_t^\varepsilon > 0$ for all $t \in (0, 1)$ and moreover $j > 0$. Hence it is easily checked any limit of discrete gradients in L^2 w.r.t. the measure $\hat{\rho}_t^\varepsilon Jm(dx, dy) = \hat{\rho}_t^\varepsilon(x, y) j(y - x) dx dy$ coincides again a.e. with a discrete gradient. Thus we have $w_t^\varepsilon = \bar{\nabla} \psi_t^\varepsilon$ a.e. for a suitable function $\psi^\varepsilon : (0, 1) \times \mathbb{R}^d \rightarrow \mathbb{R}$. Now observe that $(\rho^\varepsilon, \psi^\varepsilon) \in \mathcal{CE}'_1(\rho_0^\varepsilon, \rho_1^\varepsilon)$ and

$$\begin{aligned} L^\varepsilon &:= \int_0^1 \sqrt{\mathcal{A}'(\rho_t^\varepsilon, \psi_t^\varepsilon)} dt = \int_0^1 \sqrt{\mathcal{A}(\mu_t^\varepsilon, \tilde{\nu}_t^\varepsilon)} dt \\ &\leq \int_0^1 \sqrt{\mathcal{A}(\mu_t, \nu_t)} dt = \mathcal{W}(\bar{\mu}_0, \bar{\mu}_1), \end{aligned}$$

where we have used Proposition 2.8 in the second line. Finally we concatenate the three curves $(\rho^{0,\varepsilon}, \psi^{0,\varepsilon})$, $(\rho^\varepsilon, \psi^\varepsilon)$ and $(\rho^{1,\varepsilon}, \psi^{1,\varepsilon})$ to obtain a curve $(\tilde{\rho}^\varepsilon, \tilde{\psi}^\varepsilon) \in \mathcal{CE}'_{1+2\varepsilon}(\bar{\rho}_0, \bar{\rho}_1)$ which satisfies

$$\begin{aligned} \int_0^{1+2\varepsilon} \sqrt{\mathcal{A}'(\tilde{\rho}_t^\varepsilon, \tilde{\psi}_t^\varepsilon)} dt &= L^{0,\varepsilon} + L^\varepsilon + L^{1,\varepsilon} \\ &\leq \mathcal{W}(\bar{\mu}_0, \bar{\mu}_1) + \varepsilon(\mathcal{I}(\bar{\mu}_0) + \mathcal{I}(\bar{\mu}_1)). \end{aligned}$$

Letting ε go to zero now yields the claim. \square

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